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LETTER TO THE EDITOR

Quantum Ising model on a quasiperiodic lattice

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Abstract. We consider a quantum Ising model where the exchange couplings and the transverse fields follow a quasiperiodic two-valued sequence. The model is shown to exhibit an Ising-like critical point. The spectrum of the critical Hamiltonian is in accordance with the predictions of conformal theory.

Since the experimental discovery of quasicrystals there has been a growing interest in studying this new state of matter [1]. A great number of theoretical papers have been devoted to explaining the stability and the unusual properties of this phase [2–4]. The magnetic phase transition on quasiperiodic lattices is also the object of active investigation. Most of the results, however, have been achieved on one-dimensional models [5–7] where the phase transition takes place at zero temperature. In higher dimensions only a few results are available. Universality seems to hold for quasiperiodic topology too, if the interaction does not depend on the length of the bonds [8]. The situation is more complicated when the interaction is bond dependent and the quasiperiodically modulated interaction plays the role of a non-periodic external field. In this case it depends on the type of lattice whether this field is irrelevant [9, 10] or whether it acts more likely as a random field and washes the phase transition completely out [11].

In this letter we investigate a quantum Ising model where the strength of the couplings follows a two-valued quasiperiodic sequence. The quantum mechanical phase transition of this system is supposed to be equivalent to the critical behaviour of some two-dimensional classical layered Ising model [12]. To our knowledge this is the first exact investigation of a model with non-trivial phase transition where the interaction is quasiperiodically modulated.

Before writing down the Hamiltonian of the model let us define the position of the atoms on the lattice in the following way:

$$x_j = j + [(j+1)/\omega](\lambda - 1) \tag{1}$$

where $\lambda > 0$, $\omega \geqslant 1$ is an irrational number and $[x]$ denotes the integer part of x. The series of lattice spacings:

$$\mu_j = x_j - x_{j-1} \tag{2}$$

is a two-valued (λ and 1) quasiperiodic sequence. If $\omega = p/q$ is a rational number expressed with p and q coprimes then the series (2) is periodic with period p, while for irrational ω the series is non-periodic. The Fibonacci lattice is characterised by $\omega = \tau$, where $\tau = (1+\sqrt{5})/2$ is the golden mean.

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Now we define the model on this lattice by the Hamiltonian:

$$H = -\sum_{i=1}^{N} \mu_i \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{N} \mu_i \sigma_i^z \tag{3}$$

where σ_i^x and σ_i^z are Pauli matrices at site i. The boundary condition (BC) is specified as $\sigma_{N+1}^x = g\sigma_1^x$, with $g = 1, 0$ and -1 for periodic, free and antiperiodic BC, respectively.

To solve (3) we proceed in the well known fashion [13]. First, the Hamiltonian is expressed in terms of fermion creation and annihilation operators, then the resulting quadratic form is diagonalised by a canonical transformation. The diagonal Hamiltonian assumes the form:

$$H = \sum_k \Lambda_k (\eta_k^+ \eta_k - \tfrac{1}{2}) \tag{4}$$

where η_k^+ and η_k are fermion creation and annihilation operators. The energy of the modes may be obtained from the solution of the following eigenvalue problem:

$$\Phi_k^{l-1} h\mu_{l-1}^2 + \Phi_k^l(\mu_{l-1}^2 + h^2\mu_l^2) + \Phi_k^{l+1} h\mu_l^2 = \tfrac{1}{4}\Lambda_k^2 \Phi_k^l \tag{5}$$

for $l = 2, 3, \ldots, N-1$. The remaining two equations depend on the form of the BC:

$$\Phi_k^N(-wh)\mu_N^2 + \Phi_k^1(g^2\mu_N^2 + h^2\mu_1^2) + \Phi_k^2 h\mu_1^2 = \tfrac{1}{4}\Lambda_k^2\Phi_k^1 \tag{5a}$$

$$\Phi_k^{N-1} h\mu_{N-1}^2 + \Phi_k^N(\mu_{N-1}^2 + h^2\mu_N^2) + \Phi_k^1(-wh)\mu_N^2 = \tfrac{1}{4}\Lambda_k^2\Phi_k^N \tag{5b}$$

where $w = g\exp(\mathrm{i}\pi N_c)$ and $N_c = \Sigma\, \eta_k^+\eta_k$ is the number of fermions. The eigenvalue problem ($5a$, b) may be written in the matrix form:

$$G\Phi_k = \tfrac{1}{4}\Lambda_k^2 \Phi_k. \tag{6}$$

First we solve the bulk problem in (5) and then turn to study the boundary effects.

It is easy to see that, at $h = 1$,

$$\Phi_0^l = C_0 \sin(\pi l) \tag{7}$$

is a solution of (5) with zero eigenvalue. This soft mode drives the phase transition which takes place at $h = 1$ independently of λ and ω. Now we try to determine the energy of the low excited modes near the critical point by choosing the eigenvectors in the following form:

$$\Phi_k^l = C\sin(\varphi_l)(1 + \Delta\alpha_l) \tag{8}$$

where

$$\varphi_l = (\pi - k)l + k\beta_l + \vartheta \tag{9}$$

$\Delta = h - 1 \ll 1$, $k \ll 1$. The α_l and β_l parameters are position dependent, but do not depend on k and Δ in leading order. In the following we show that (8) is an eigenvector of G to linear order. Let us insert (8) into (6) and denote the resulting vector by $\psi_k = G\phi_k$. Retaining terms up to second order we obtain

$$\begin{aligned}\Psi_k^l = \Phi_k^l\{&\Delta(A_{l-1} - A_l) + k\cot(\varphi_l)(B_{l-1} - B_l) - \Delta^2 A_l + \tfrac{1}{2}k^2(b_{l-1}B_{l-1} + b_l B_l) \\ &+ \Delta k\cot(\varphi_l)[a_{l-1}B_{l-1} - (a_l + 2)B_l]\} + O_3.\end{aligned} \tag{10}$$

Here

$$a_l = \alpha_{l+1} - \alpha_l - 1 \qquad b_l = \beta_{l+1} - \beta_l - 1 \tag{11a}$$

and

$$A_l = a_l\mu_l^2 \qquad B_l = b_l\mu_l^2 \tag{11b}$$

while the omitted terms contain expressions with k^3, Δ^3, $k^2\Delta$, $k\Delta^2$, ... It is easy to observe from (10) that ϕ_k in (8) is an eigenvector of G to linear order with zero eigenvalue if the conditions

$$A_l = A = \text{constant} \qquad B_l = \text{constant} \tag{12}$$

are fulfilled (in leading order). To satisfy these equations we look for other relations between the a_l and b_l parameters. First, note that by symmetry reasons these parameters may have only two values—a_λ, a_1 and b_λ, b_1—depending on the value of the lth bond. On the other hand, according to the definition (11a):

$$(1/N)\sum_l (a_l+1) = (a_\lambda+1)n_\lambda + (a_1+1)n_1 = 0 \tag{13}$$

where n_λ and n_1 are the density of the λ and 1 bonds, respectively:

$$n_\lambda = 1/\omega \qquad n_1 = 1 - 1/\omega. \tag{14}$$

These relations (13) and (14) are also true for the b_l coefficients, so $b_l = a_l$. Equations (12) are trivially fulfilled for the same type of bonds while for different bonds one obtains using (11b), (13) and (14):

$$A = B = -\lambda^2\omega/[\lambda^2(\omega-1)+1]. \tag{15}$$

Thus we have determined the eigenvectors up to linear order. The next step is to calculate the eigenvalues up to second order by using these eigenvectors. The eigenvalues are given as

$$\Lambda_k^2/4 = \left(\sum_l \Psi_k^l \Phi_k^l\right)\left(\sum_l \Phi_k^l \Phi_k^l\right)^{-1}. \tag{16}$$

Using (10) this expression is easy to evaluate with the result:

$$\Lambda_k^2/4 = -A(\Delta^2+k^2) + O_3. \tag{17}$$

Here we made use of the relation:

$$\left(\sum_l \Phi_k^l \Phi_k^l b_l\right)\left(\sum_l \Phi_k^l \Phi_k^l\right)^{-1} = -1 \tag{18}$$

and the Δk term vanishes, while

$$\left(\sum_l \sin(2\varphi_l)\right)\left(\sum_l \sin^2(\varphi_l)\right)^{-1} = 0. \tag{19}$$

Thus, according to (17), the dispersion relation for the low excited states is given by

$$\Lambda_k = \xi(\lambda,\omega)(\Delta^2+k^2)^{1/2} \tag{20}$$

where

$$\xi(\lambda,\omega) = 2\lambda\{\omega[\lambda^2(\omega-1)+1]^{-1}\}^{1/2} \tag{21}$$

is the sound velocity. Relation (20) is the same as for the quantum Ising model [14, 15]. Since the critical behaviour of the system is determined by these low-energy excitations we can conclude that our model belongs to the Ising universality class for all values of λ and ω. This is the main result of our letter. It is easy to draw consequences of equation (20). The dispersion relation at the critical point is linear: $\Lambda_k(h=1) = \xi(\lambda,\omega)k$. At this point the energy gap linearly vanishes:

$$E_1 - E_0 = \xi(\lambda,\omega)(h-1). \tag{22}$$

Thus the correlation length exponent is $\nu = 1$. Furthermore the specific heat has a logarithmic singularity at $h = 1$.

Now we turn to analyse the effect of boundary conditions. For periodic and antiperiodic BC equations (5*a*, *b*) are satisfied if

$$\Phi_k^0 + w\Phi_k^N = 0 \qquad \Phi_k^{N+1} + w\Phi_k^1 = 0. \tag{23}$$

For small k and Δ these equations are fulfilled with the functions (8) if

$$\cos[(\pi - k)N] = -w \qquad \sin[(\pi - k)N] = 0. \tag{24}$$

These relations are the same as those of the quantum Ising model [15]. Thus we can borrow the results. The possible k values are from two sets:

$$\mu = \{\pm(\pi/N)(2m-1)\} \qquad \nu = \{0, \pm(\pi/N)2m\} \tag{25}$$

where $m = 1, 2, \ldots,$ and $m \ll N$. The allowed values depend on the parity of N_c and on the form of the BC:

(*a*) N_c even

$$k_P \in \mu \qquad k_{AP} \in \nu$$

(*b*) N_c odd

$$k_P \in \nu \qquad k_{AP} \in \mu.$$

(The subscripts P and AP refer to periodic and antiperiodic BC, respectively.)

Since the dispersion relation (20) and the allowed k values for our model and for the quantum Ising model are the same the two models also share the tower-like structure of the spectrum at the critical point in the finite-size scaling limit [16]:

$$E_i - E_0 = (2\pi/N)\xi(\lambda, \omega)(x + n + n'). \tag{26}$$

Here E_0 and E_i denote the energy of the ground state and the ith excited state, x is the anomalous dimension of a primary operator and n and n' are non-negative integers. For periodic BC in the energy sector (N_c is even) $x_e = 2$. In the magnetisation sector (N_c is odd) x_m measures the difference in the ground-state energies for periodic and antiperiodic BC. Thus to determine x_m one needs information about the complete spectrum which is not available in our calculation. Therefore we performed numerical calculations for large systems ($N \approx 300$). Our results confirm, with high accuracy, the suggestion that $x_m = \frac{1}{8}$, which is as expected from universality.

To summarise, we have solved exactly a quantum Ising model where the couplings follow the quasiperiodic sequence (2). The Ising-like phase transition is preserved for all values of λ and ω. Furthermore the low-lying excitations at the critical point have a conformal tower structure.

Due to the way of derivation our results remain valid for other similar problems. First, we mention the case of periodic structures, which corresponds to rational ω values. Our earlier solution of the quantum Ising model with a staggered interaction [17] is recovered in (20) for $\omega = 2$. Another possible extension of the results is to the problem of an Ising model with two-valued random bonds where the density of the bonds is given by (14). According to (20), for this special random system the Ising phase transition is universally preserved, similar to two-dimensional Ising models with layered impurities where the interaction energies follow a Dirac delta distribution [12].

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